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Quantum Evolution in Fluctuating Backgrounds: Nonideal Clocks and Foam-like Spacetimes

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I characterize good clocks, which are naturally subject to fluctuations, in statistical terms, obtain the master equation that governs the evolution of quantum systems according to these clocks, and find its general solution. This master equation is diffusive and produces loss of coherence. Moreover, real clocks can be described in terms of effective interactions that are nonlocal in time. Alternatively, they can be modeled by an effective thermal bath coupled to the system. I also study some aspects concerning the evolution of quantum low-energy fields in a foamlike spacetime, with involved topology at the Planck scale but with a smooth metric structure at large length scales. This foamlike structure of spacetime may show up in low-energy physics through loss of quantum coherence and mode-dependent energy shifts, for instance, which might be observable. Spacetime foam introduces nonlocal interactions that can be modeled by a quantum bath, and low-energy fields evolve according to a master equation that displays such effects. These evolution laws are similar to those for quantum mechanical systems evolving according to good nonideal clocks, although the underlying Hamiltonian structure in this case establishes some differences among both scenarios.

KEY WORDS: spacetime foam; quantum spacetime; fluctuations; imperfect clocks in quantum mechanics.

1. INTRODUCTION

Any real clock is inevitably subject to quantum fluctuations, which introduce uncertainties in the equations of motion. For instance, it has been shown that the finite mass and size of the clock impose limitations in the measurement of spacetime distances in the framework of general relativity (Salecker and Wigner, 1958; Wigner, 1957). Some considerations have also been made on the role of quantum clocks in the context of quantum cosmology (Hartle, 1998; Unruh and Wald, 1989). Simple models for quantum clocks have been proposed, and the quantum evolution of a system according to a quantum clock suitably coupled with it has also been studied (Aharanov *et al.*, 1998; Hartle, 1988; Page and

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Wootters, 1983; Peres, 1980; Salecker and Wigner, 1958; Unruh and Wald, 1989; Wigner, 1957). The general conclusion is that the system becomes more and more perturbed as the resolution of the clock is improved. Even more, quantum gravity may well imply the existence of an absolute limit, the Planck scale, to the accuracy of spacetime–distance measurements and, in particular, to clock synchronization (for a review, see, e.g., Ref. Garay, 1995), with possible effects in the low-energy regime (Amelino-Camelia *et al.*, 1998; Ellis *et al.*, 1984; Garay, 1998a,b; Hawking, 1982), as discussed below.

It follows that any quantum clock that we could possibly build would lead to uncertainties and errors. These quantum errors, however, are not the only source of randomness in the measure of time. Real clocks are also subject to classical imperfections, small errors, that can only be dealt with statistically. For instance, an unavoidable classical source of stochasticity is temperature, which will introduce thermal fluctuations in the behavior of real clocks. Although this is not necessarily the most important source of errors in modern day atomic clocks, it is nonetheless always present to some extent. In other words, the third law of thermodynamics forbids the existence of ideal clocks. Even at zero-temperature, the quantum vacuum fluctuations of quantum field theory make propagating physical systems (real clocks among them) suffer a cold diffusion and consequently a need for a stochastic description of their evolution (Gour and Sriramkumar, 1999). We are then bound to use real physical clocks and rely on their readouts when measuring the evolution of a quantum system.

The stochastic nature of nonideal clocks, naturally leads to the conclusion that the evolution according to the readouts of a real clock is nonunitary. In other words, the use of real clocks induces loss of coherence in most physical quantum states, as we will explicitly show. A similar although more complicated situation appears within the context of quantum gravity: low-energy fields may lose coherence because of the unavoidable quantum fluctuations of spacetime. Let us briefly review how this may occur.

A quantum uncertainty in the position of a particle implies an uncertainty in its momentum and, therefore, due to the gravity–energy universal interaction, would also imply an uncertainty in the geometry, which in turn would introduce an additional uncertainty in position of the particle. The geometry would thus be subject to quantum fluctuations that would constitute the spacetime foam and that should be of the same order as the geometry itself at the Planck scale. This would give rise to a minimum length (Garay, 1995) beyond which the geometrical properties of spacetime would be lost, while on larger scales it would look smooth and with a well-defined metric structure. The key ingredients for the appearance of this minimum length are quantum mechanics, special relativity, which is essential for the unification of all kinds of energy via the finiteness of the speed of light, and a theory of gravity, i.e., a theory that accounts for the active response of spacetime to the presence of energy. Thus, the existence of a lower bound to any output

of a position measurement seems to be a model-independent feature of quantum gravity (Garay, 1995).

More than a century ago, Riemann (1873) already noticed that "Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena." In the middle of this century, Weyl (1949) took these ideas a bit further and envisaged topological structures of ever-increasing complexity as possible constituents of the physical description of surfaces. Few years later, Wheeler (1957) described this topological complexity of spacetime at small length scales as the foam-like structure of spacetime. According to Wheeler, at the Planck scale, the fluctuations of the geometry are so large and involve so large energy densities that gravitational collapse should be continuously being done and undone at that scale. Because of this perpetuity and ubiquity of Planck scale gravitational collapse, it should dominate Planck scale physics. In this continuously changing scenario, it seems natural to accept that the topology of spacetime is also subject to quantum fluctuations. Furthermore, from the functional integration point of view, in quantum gravity all histories contribute and, among them, there seems unnatural not to consider nontrivial topologies as one considers nontrivial geometries (Misner, 1960; Wheeler, 1957) (see, however, Ref. DeWitt, 1984).

The quantum structure of spacetime would be relevant at energies close to Planck scale and one could expect that the quantum gravitational virtual processes that constitute the spacetime foam could not be described without knowing the details of the theory of quantum gravity. However, the gravitational nature of spacetime fluctuations provides a mechanism for studying the effects of these virtual processes in the low-energy physics. Indeed, virtual gravitational collapse and topology change would forbid a proper definition of time at the Planck scale. In general, these spacetime fluctuations, in which the asymptotically time-like Hamiltonian vector fields vanish, are associated with infinite redshift surfaces and, consequently, these small spacetime regions would behave as magnifiers of Planck length scales transforming them into low-energy modes as seen from outside the fluctuations (Padmanabhan, 1999). Therefore, spacetime foam and the related lower bound to spacetime uncertainties would leave their imprint, which may be not too small, in low-energy physics (see, e.g., Garay, 1999 and references therein). Also, low-energy experiments would effectively suffer a nonvanishing uncertainty coming from this lack of resolution in spacetime measurements. In this situation, loss of quantum coherence would be almost unavoidable (Hawking, 1982; Hawking *et al.*, 1979, 1980). This conclusion was based in part on the thermal character of the emission predicted for evaporating black holes (Hawking, 1976). If loss of coherence occurs in macroscopic black holes, it seems reasonable

to conclude that the small black holes that are continuously being created and annihilated everywhere within spacetime foam will also induce loss of quantum coherence (Hawking, 1976, 1982). Here, I will describe a physical low-energy effective model (Garay, 1998a,b, 1999) that realizes this conjecture. Furthermore, related to this issue, spacetime foam might produce frequency-dependent energy shifts (Garay, 1998a,b, 1999) that would slightly alter the dispersion relations for the different low-energy fields.

Throughout this paper, I will set $\hbar = c = 1$, so that the only dimensionfull constant will be Planck's length $\ell_* = \sqrt{G}$, G being Newton's constant.

2. NONIDEAL CLOCKS

In this section, we will study, within the context of the standard quantum theory, the evolution of an arbitrary system according to a real nonideal clock (Egusquiza *et al.*, 1999), i.e., we will be concerned with the readings of time that it provides. As stated before, these readings will undergo errors, which will be described by a stochastic process. In what follows we shall not delve further into the source of stochasticity, but assume a phenomenological description of it.

Let us imagine a large ensemble of identical systems, prepare one of them in a given initial state at initial clock time $t = 0$, and then measure the state of that system at clock time t. If we repeat this procedure for all the systems in the ensemble, the result will be a probability distribution for the possible outcomes, its dispersion partially being a consequence of the lack of knowledge of the precise ideal time that has elapsed.

A real clock will be a system with a degree of freedom t that closely follows the ideal time parameter $t_{\rm i}$, i.e., $t_{\rm i} = t + \Delta(t)$, where $\Delta(t)$ is the error at the real clock time t. Given any real clock, its characteristics will be encoded in the probability functional distribution for the continuous stochastic processes $\Delta(t)$ (Gardiner, 1985; Van Kampen, 1981) of clock errors, $\mathcal{P}[\Delta(t)]$, which must satisfy appropriate conditions, so that it can be regarded as a good clock.

2.1. Good Clock Conditions

A first property is that Galilean causality should be preserved, i.e., that causally related events should always be properly ordered in clock time as well, which implies that $t_{\rm i}(t') > t_{\rm i}(t)$ for every $t' > t$. In terms of the derivative $\alpha(t) = d\Delta(t)/dt$ of the stochastic process $\Delta(t)$, we can state this condition as requiring that, for any realization of the stochastic sequence, $\alpha(t) > -1$.

A second condition that we would require good clocks to fulfill is that the expectation value of relative errors (per unit real time), determined by the stochastic process $\alpha(t)$, be zero, i.e., $\langle \alpha(t) \rangle = 0$ for all t. If this were not the case, the clock would either systematically go fast or slow down, and a redefinition through this

systematic drift would provide us with a well-centered clock. Consequently, the expectation value for the absolute errors $\Delta(t)$ will be constant. Furthermore, since $t = 0$ will be the time at which the systems whose evolution we are studying are prepared, $\Delta(0)$ will not be stochastic and, without loss of generality, will be set to zero by a simple translation of the origin of time, so that $\langle \Delta(t) \rangle = 0$.

Furthermore, a good clock should always behave in the same way (in a statistical sense). We can say that the clock behaves consistently in time as a good one if those relative errors $\alpha(t)$ are statistically stationary, i.e., the probability functional distribution $\mathcal{P}[\alpha(t)]$ for the process of relative errors $\alpha(t)$ (which can be obtained from $P[\Delta(t)]$, and vice versa) must not be affected by global shifts $t \to t + t_0$ of the readout of the clock. Note that the stochastic process $\Delta(t)$ need not be stationary, despite the stationarity of the process $\alpha(t)$.

The one-point probability distribution function for the variables $\alpha(t)$ should be highly concentrated around the zero mean, if the clock is to behave nicely. Even more, it is to be expected for clocks with small errors that all the higher-order cumulants be much smaller than the correlation, which, in turn, should also be bounded by a small number, i.e.,

$$\langle \alpha(t)\alpha(t-\tau) \rangle \equiv c(\tau) \leq c(0) \ll 1,$$

where $c(\tau) = c(-\tau)$. The correlation for the sequence of absolute errors $\Delta(t)$ can then be easily obtained and has the form

$$\langle \Delta(t)\Delta(t') \rangle = \int_0^t dt_1 \int_0^{t'} dt_2 \, c(t_1 - t_2).$$

The correlation time ϑ for the stochastic process $\alpha(t)$ is given by

$$\vartheta = \int_0^\infty \frac{c(\tau)}{c(0)}.$$

We will introduce a new parameter κ with dimensions of time, defined as $\kappa^2 = c(0)\vartheta^2$. This comes about because, when the errors of the clock have a thermal origin, κ^2 is proportional to the temperature, and independent of ϑ. In general, the good clock conditions imply $\kappa \ll \vartheta$. As we shall see, ϑ cannot be arbitrarily large, and, therefore, the ideal clock limit is given by $\kappa \to 0$.

Until now we have discussed general properties that a good clock must fulfill, regardless of the physical system under study. In addition to these properties, a good clock must have enough precision in order to measure the evolution of the specific system, which imposes further restrictions on the clock. On the one hand, the characteristic evolution time ζ of the system must be much larger than the correlation time ϑ of the clock. On the other hand, the leading term in the asymptotic expansion of the variance $\langle \Delta(t)^2 \rangle$ for large t is of the form $\kappa^2(k\varepsilon/\vartheta)$, which means that, after a certain period of time, the absolute errors can be too large. The maximum admissible standard deviation in $\Delta(t)$ must be at most of

the same order as ζ. Then the period of applicability of the clock to the system under study, i.e., the period of clock time during which the errors of the clock are smaller than the characteristic evolution time of the system is approximately equal to $\zeta^2\vartheta/\kappa^2$. For a good clock, $\kappa \ll \vartheta \ll \zeta$, as we have seen, so that the period of applicability is much larger than the characteristic evolution time ζ.

2.2. Evolution Laws

We shall now obtain the evolution equation for the density matrix of an arbitrary quantum system in terms of the clock time t.

Let H be the time-independent Hamiltonian of the system so that, in terms of the ideal time $t_{\rm i}$, its action has the form

$$S = \int dt_{\rm i} \left[p\frac{dq}{dt_{\rm i}} - H \right].$$

For any given realization of the stochastic process $\alpha(t)$ that characterizes a good clock, the relation between the ideal time $t_{\rm i}$ and the real time t is given by $t_{\rm i} = t + \int_0^t dt'\alpha(t')$. Thus, a straightforward change of the time integration variable yields the action S_α for each realization $\alpha(t)$:

$$S_\alpha = \int dt\{p\dot{q} - [1+\alpha(t)]H\} = S - \int dt\,\alpha(t)H,$$

where the overdot denotes derivative with respect to the real time t. The density matrix $\rho_\alpha(t)$ for an arbitrary quantum system and for a given realization of the stochastic process $\alpha(t)$, can be obtained from the initial density matrix $\rho(0)$ by means of a linear operator called the superscattering operator $\$_\alpha(t)$:

$$\rho_\alpha(t) = \$_\alpha(t)\cdot\rho(0), \tag{1}$$

where the superscattering operator has the form

$$\$_\alpha(t) = \int \mathcal{D}Q\mathcal{D}Q' e^{iS_\alpha[Q;t]-iS_\alpha[Q';t]},$$

with $Q \equiv (q,\, p)$.

The average of the density matrix $\rho_\alpha(t)$ can be regarded as the density matrix of the system $\rho(t)$ at the clock time t. Since the initial density matrix is independent of α, its evolution is determined by the superscattering operator \$

$$\$(t) = \int \mathcal{D}\alpha\,\mathcal{P}[\alpha]\$_\alpha(t), \qquad \rho(t) = \$(t)\cdot\rho(0). \tag{2}$$

Under the good clock approximation $\kappa \ll \vartheta$, only the two-point correlation function $c(\tau)$ is relevant, so that we can approximate $\mathcal{P}[\alpha(t)]$ by a stationary Gaussian probability functional with zero mean and correlation given by the correlation $c(\tau)$ of $\mathcal{P}[\alpha(t)]$. Although this Gaussian approximation assigns a

nonvanishing probability to $\alpha(t) < -1$, this probability will be negligibly small since, for good clocks, $c(t) \ll 1$. Thus the Gaussian approximation to good clocks fulfills the Galilean causality condition for all practical purposes. The integration over $\alpha(t)$ is then easily performed to obtain the influence action $\mathcal{W}$

$$\mathcal{W}[Q, Q'; t] = -\frac{1}{2}\int_0^t ds \int_0^s ds' \{H[Q(s)] - H[Q'(s)]\} c(s - s') \{H[Q(s')] - H[Q'(s')]\},$$

in terms of which the superscattering operator has the form

$$\$(t) = \int \mathcal{D}Q\mathcal{D}Q'\, e^{iS[Q;t] - iS[Q';t]}\, e^{\mathcal{W}[Q, Q';t]},$$

This superscattering operator corresponds to the evolution of a system with a free Hamiltonian H coupled with a classical noise source $\alpha(t)$, with a probability functional distribution $\mathcal{P}[\alpha(t)]$, via the interaction Hamiltonian $\alpha(t)H$. We see that there is no dissipative term there as could be expected from the fact that the noise source is classical (Feynman and Hibbs, 1965; Feynman and Vernon, 1963). Moreover, as the interaction term is proportional to H, there is no response of the system to the outside noise, which means that the associated impedance is infinite (Callen and Welton, 1951; Gardiner, 1991; Mandel and Wolf, 1995).

Therefore, we see that the effect of using good real clocks for studying the evolution of a quantum system is the appearance of an effective interaction term in the action integral which is bilocal in time. This can be understood as the first term in a multilocal expansion, which corresponds to the weak-field expansion of the probability functional around the Gaussian term. This nonlocality in time admits a simple interpretation: correlations between relative errors at different instants of clock-time can be understood as correlations between clock-time flows at those clock instants. The clock-time flow of the system is governed by the Hamiltonian and, therefore, the correlation of relative errors induces an effective interaction term, generically multilocal, that relates the Hamiltonians at different clock instants.

From the form of the influence action or equivalently from Eq. (2), it is not difficult to see that, in the Markov approximation and provided that the system evolves for a time smaller than the period of applicability of the clock, the density matrix $\rho(t)$ satisfies a master equation that contains a diffusion term. Indeed, the differential version of the scattering equation (2) can be found by differentiating such equation with respect to clock-time. In order to do that we will first analyze the corresponding differential equation for $\rho_\alpha(t)$, and then we will average over all possible realizations of the stochastic process α as required in Eq. (2). The differential equation satisfied by $\rho_\alpha(t)$ can be obtained by differentiating Eq. (1)

with respect to clock time t:

$$\dot{\rho}_\alpha(t) = -i(1+\alpha(t))[H, \rho_\alpha(t)].$$

Let us now transform to the interaction picture in which the density matrix has the form

$$\rho_\alpha^{\mathrm{I}}(t) = e^{iHt}\rho_\alpha(t)\, e^{-iHt}.$$

Notice that the interaction term $\alpha(t)H$ has the same form in both pictures because it is proportional to the free Hamiltonian H. Integrating the resulting equation between 0 and t, and reintroducing the result for ρ_α^{I}, we obtain the following integro-differential equation:

$$\dot{\rho}_\alpha^{\mathrm{I}}(t) = -i\alpha(t)\big[H, \rho_\alpha^{\mathrm{I}}(0)\big] - \int_0^t dt'\alpha(t)\alpha(t')\big[H, \big[H, \rho_\alpha^{\mathrm{I}}(t')\big]\big].$$

In order to find the evolution equation in the clock-time t, we have to average this equation over all possible realizations $\alpha(t)$ of the stochastic process with the functional weight $\mathcal{P}[\alpha(t)]$. The average of the density matrix $\rho_\alpha^{\mathrm{I}}(t)$ will be denoted by $\rho^{\mathrm{I}}(t)$ and can be regarded as the interaction-picture density matrix of the system at clock-time t.

At the real time $t = 0$, we impose the initial condition $\rho^{\mathrm{I}}(0) = \rho_\alpha^{\mathrm{I}}(0) = \rho(0)$. Additionally, for a good clock, $\langle\alpha(t)\rangle = 0$, as already discussed, and, as a consequence, the average of the linear term in $\alpha(t)$ vanishes. Furthermore, the clock-time derivative ∂_t and the average over $\alpha(t')$ commute because $\mathcal{P}[\alpha(t')]$ is stationary. Finally, the density matrix $\rho_\alpha^{\mathrm{I}}(t')$ can be expanded in powers of $\alpha(t')$. Then the average of the integro-differential equation for the density matrix ρ^{I} yields

$$\dot{\rho}^{\mathrm{I}}(t) = -\int_0^t d\tau\, c(\tau)\big[H, \big[H, \rho^{\mathrm{I}}(t-\tau)\big]\big] + O(\langle\alpha^3\rangle),$$

where we have performed a change of the integration variable from t' to $\tau = t - t'$ and have introduced the correlation function $c(\tau)$ for the stochastic process $\alpha(t)$.

For a good clock, the higher-order terms in α can be seen to be much smaller than the $c(\tau)$ term by a factor $(\kappa/\zeta)^2 \ll 1$, provided that the system evolves for a time smaller than the period of applicability of the clock. Since $\zeta \gg \vartheta$, the system does not evolve significantly within a correlation time, and we can substitute $\rho^{\mathrm{I}}(t-\tau)$ by $\rho^{\mathrm{I}}(t)$. This is the so-called Markov approximation. The process $\Delta(t)$ will not be Markovian in general and there is no reason for requiring that the process $\alpha(t)$ has this property either. However, and even though the Markov approximation refers to the system and not to the clock itself, it renders the possible non-Markovian character of the clock irrelevant. Furthermore, for evolution times t much larger than the correlation time ϑ, we can take the upper integration limit to infinity.

The resulting master equation, once we go back to the Schrödinger picture, can be written as

$$\dot{\rho}(t) = -i\big[H, \rho(t)\big] - (\kappa^2/\vartheta)\big[H, \big[H, \rho(t)\big]\big].$$

Notice that the accuracy of the clock appears in the master equation through the parameters κ and ϑ and that, in the ideal clock limit, $\kappa \to 0$, the unitary von Neumann equation is recovered. We should also point out that this master equation is not a truncation of the BBGKY hierarchy (Huang, 1987), and that irreversibility appears because the errors of the clock cannot be eliminated once we have started using it.

In the Gaussian approximation, there is essentially only one good clock for which $\alpha(t)$ is Markovian, the Ornstein-Uhlenbeck process (Gardiner, 1985; Van Kampen, 1981). In this case, the correlation function for $\alpha(t)$ in the stationary regime is $c(\tau) = (\kappa/\vartheta)^2 e^{-|\tau|/\vartheta}$. Since the possible non-Markovian character of the clock does not influence the time evolution of the system (provided that the condition $\zeta \ll \vartheta$ is satisfied, as happens for good clocks), the Ornstein-Uhlenbeck clock is generic in what concerns the evolution of quantum systems according to real clocks.

2.3. Effective Thermal Bath

A real clock can be effectively modeled by a thermal bath, with a temperature T_b to be determined, coupled to the system.

Let $H + H_\mathrm{int} + H_\mathrm{b}$ be the total Hamiltonian, where H is the free Hamiltonian of the system and H_b is the Hamiltonian of a bath that will be represented by a collection of harmonic oscillators (Gardiner, 1991; Mandel and Wolf, 1995). The interaction Hamiltonian will be of the form $H_\mathrm{int} = \xi H$, where the noise operator ξ is given by

$$\xi(t) = \frac{i}{\sqrt{2\pi}} \int_0^\infty d\omega \chi(\omega)[a^\dagger(\omega)\, e^{i\omega t} - a(\omega)\, e^{-i\omega t}].$$

In this expression, a and $a^\dagger$ are, respectively, the annihilation and creation operators associated with the bath, and $\chi(\omega)$ is a real function, to be determined, that represents the coupling between the system and the bath for each frequency ω.

Identifying, in the classical noise limit, the classical correlation function of the bath with $c(\tau)$, the suitable coupling between the system and the bath is given by the spectral density of fluctuations of the clock:

$$T_\mathrm{b}\chi(\omega)^2 = \int_0^\infty d\tau\, c(\tau)\, \cos(\omega\tau).$$

With this choice, the master equation for evolution according to real clocks is identical to the master equation for the system obtained by tracing over the effective bath.

To go beyond the classical noise limit requires the introduction of the usual quadratic dissipation term in the influence functional (Feynman and Hibbs, 1965; Feynman and Vernon, 1963). However, the peculiar coupling to the energy ξH, which is quite different from the usual coupling to the position or the momentum of the system, implies that this term does not produce dissipation in the equations of motion: the fluctuation–dissipation theorem, which reflects the microscopic structure of the bath, is thus fulfilled, but there is no dissipation.

2.4. Decoherence

The master equation contains a diffusion term and will, therefore, lead to a loss of coherence (Giulini *et al.*, 1996). However, this loss depends on the initial state. In other words, there exists a pointer basis (Zurek, 1981, 1982), so that any density matrix that is diagonal in this specific basis will not be affected by the diffusion term, while any other will approach a diagonal density matrix. The stochastic perturbation $\alpha(t)H$ is obviously diagonal in the basis of eigenstates $\{|n\rangle\}$ of the Hamiltonian, which is, therefore, the pointer basis: the interaction term cannot induce any transition between different energy levels ω_n.

The components of the density matrix in this basis are $\rho_{nm} = \langle n|\rho|m\rangle$. The master equation can be solved exactly, its general solution being

$$\rho_{nm}(t) = \rho_{nm}(0)\, e^{-i\omega_{nm}t}\, e^{-(\omega_{nm})^2\kappa^2 t/\vartheta},$$

where $\omega_{nm} = \omega_n - \omega_m$. The smallest energy difference ω provides the inverse of the characteristic time for the evolution of the system, $\zeta = 1/\omega$. The smallest decay constant is $\omega^2\kappa^2/\vartheta$, equal to the inverse of the period of applicability of the clock. By the end of this period, the density matrix will have been reduced to the diagonal terms and a much diminished remnant of those off-diagonal terms with slow evolution. In any case, the von Neumann entropy grows if the density matrix is not initially diagonal in the energy basis.

The effect of decoherence due to errors of real clocks does not only turn up in the quantum context. Consider, for instance, a classical particle with a definite energy E moving under a time-independent Hamiltonian H. Because of the errors of the clock, we cannot be positive about the location of the particle in its trajectory on phase space at our clock time t. Therefore, we have an increasing spread in the coordinate and conjugate momentum over the trajectory. For a generic system, this effect is codified in the classical master equation,

$$\dot{\rho} = \{H, \rho\} + (\kappa^2/\vartheta)\{H, \{H, \rho\}\},$$

where, here, $\rho(t)$ is the probability distribution on phase space in clock time. This classical master equation can be derived in a manner completely analogous to the quantum one.

For simplicity, let us study the particular example of a one-dimensional Hamiltonian motion with closed orbits, with $H = \omega J$, φ being the angle variable with period 2π conjugate to the action variable J, and ω a constant frequency characteristic of the system. The classical master equation for the probability density $\rho(\varphi, J; t)$ reads

$$\dot{\rho} = \omega \partial_\varphi \rho + (\omega^2 \kappa^2 / \vartheta) \partial_\varphi^2 \rho.$$

This diffusion equation can be exactly solved by separation of variables. The slowest decaying mode has, as before, a decay constant $\omega^2 \kappa^2 / \vartheta$.

In the case of one particle that is released with energy E and initial angle φ_0, the probability distribution spreads out over the corresponding connected component of the energy shell, and tends to $\delta(J - E/\omega)/2\pi$ as clock time grows. As we can see, the information about the φ variable is washed out by the errors in our clock, which is precisely the information that is not available in the quantum case: if J is completely known for a given quantum state, the indeterminacy in its conjugate variable will be infinite, the situation towards which the classical decoherence process tends.

Finally, it should be observed that the mechanism of decoherence is neither tracing over degrees of freedom, nor coarse graining, nor dephasing (Cooper *et al.*, 1997; Giulini *et al.*, 1996). Even though there is no integration over time introduced here by fiat, as happens in dephasing in quantum mechanics, the spread in time due to the errors of the clock has a similar effect, and produces decoherence.

3. SPACETIME FOAM

In the previous section, we have analyzed the evolution of physical systems when measured by real clocks, which are generally subject to errors and fluctuations, in contrast with ideal clocks that, although would accurately measure the time parameter that appears in the Schrödinger equation, do not exist in nature.

On the other hand, spacetime foam contains highly nontrivial topological or causal configurations, which will introduce additional features in the description of the evolution of low-energy fields as compared with topologically trivial, globally hyperbolic manifolds. The similarity with quantum systems evolving according to a nonideal clock as described above is striking. Actually, despite the different conceptual and physical origin of the fluctuations, we will see that the effects of these two systems are indeed similar, although there also exist important differences.

In order to build an effective theory that accounts for the propagation of low-energy fields in a foam-like spacetime, we will substitute the spacetime foam, in which we possibly have a minimum length because the notion of distance is not valid at such scale, by a fixed background with low-energy fields living on it. We will perform a 3+1 foliation of the effective spacetime that, for simplicity, will be regarded as flat, t denoting the time parameter and x the spatial coordinates. The

gravitational fluctuations and the minimum length present in the original spacetime foam will be modeled by means of nonlocal interactions that relate spacetime points that are sufficiently close in the effective background, where a well-defined notion of distance exists (Garay, 1998a,b, 1999) (for related ideas and a review on stochastic gravity see Ref. Hu, 1999). Furthermore, these nonlocal interactions will be described in terms of local interactions as follows. Let $\{h_i[\phi; t]\}$ be a basis of local gauge–invariant interactions at the spacetime point (x, t) made out of factors of the form $\ell_*^{2n(1+s)-4}[\phi(x, t)]^{2n}$, ϕ being the low-energy field strength of spin s. As a notational convention, each index i implies a dependence on the spatial position x by default; whenever the index i does not carry an implicit spatial dependence, it will appear underlined $\underline{i}$. Also, any contraction of indices (except for underlined ones) will entail an integral over spatial positions.

3.1. Influence Functional

The low-energy density $\rho[\phi, \varphi; t]$ at the time t in the field representation can be generally related to the density matrix at $t = 0$

$$\rho[\phi, \varphi; t] = \int D\phi'\, D\varphi' \$[\phi, \varphi; t \mid \phi', \varphi'; 0]\rho[\phi', \varphi'; 0],$$

which we will write in the compact form $\rho(t) = \$(t) \cdot \rho(0)$. Here $\$(t)$ is the propagator for the density matrix and $D\phi \equiv \prod_x \phi(x, t)$. This propagator has the form

$$\$[\phi, \varphi; t \mid \phi', \varphi'; 0] = \int \mathcal{D}\phi\, \mathcal{D}\varphi\, e^{iS_0[\phi;t]-iS_0[\varphi;t]}\, e^{\mathcal{W}[\phi,\varphi;t]},$$

where $\exp \mathcal{W}[\phi, \varphi; t]$ is the influence functional (Feynman and Hibbs, 1965; Feynman and Vernon, 1963), $\mathcal{D}\phi \equiv \prod_{x,s} \phi(x, s)$ and these path integrals are performed over paths $\phi(s)$, $\varphi(s)$ such that at the end points match the values ϕ, φ at t and ϕ', φ' at $s = 0$. The influence action $\mathcal{W}[\phi, \varphi; t]$ contains all the information about the interaction of the low-energy fields with spacetime foam. If the influence action $\mathcal{W}[\phi, \varphi; t]$ were equal to the zero, then we would have unitary evolution provided by a factorized superscattering matrix. However, $\mathcal{W}$ does not vanish in the presence of gravitational fluctuations and, in fact, the nonlocal effective interactions will be modeled by terms in $\mathcal{W}$ that follow the pattern

$$\int dt_1 \ldots dt_N \upsilon^{i_1 \ldots i_N}(t_1 \ldots t_N) h_{i_1}[\phi; t_1] \ldots h_{i_N}[\phi; t_N].$$

Here, $\upsilon^{i_1 \cdots i_N}(t_1 \ldots t_N)$ are dimensionless complex functions that vanish for relative spacetime distances larger than the length scale r of the gravitational fluctuations. If the gravitational fluctuations are smooth in the sense that they only involve trivial topologies or contain no horizons, the coefficients $\upsilon^{i_1 \cdots i_N}(t_1 \ldots t_N)$ will be N-point propagators that, as such, will have infinitely long tails and the size of

the gravitational fluctuations will be effectively infinite. In other words, we would be dealing with a local theory written in a nonstandard way. The gravitational origin of these fluctuations eliminate these long tails because of the presence of gravitational collapse and topology change.

The coefficients $\upsilon^{i_1 \dots i_N}(t_1 \dots t_N)$ can depend only on relative positions and not on the location of the gravitational fluctuation itself. The physical reason for this is conservation of energy and momentum: the fluctuations do not carry energy, momentum, or gauge charges. Thus, diffeomorphism invariance is preserved, at least at low-energy scales. One should not expect that at the Planck scale this invariance still holds. However, this violation of energy–momentum conservation is safely kept within Planck scale limits (Unruh and Wald, 1995), where the processes will no longer be Markovian.

Finally, the coefficients $\upsilon^{i_1 \dots i_N}(t_1 \dots t_N)$ will contain a factor $[e^{-S(r)/2}]^N$, $S(r)$ being the Euclidean action of the gravitational fluctuation, which is of the order $(r/\ell_*)^2$. This is just an expression of the idea that inside large fluctuations, interactions that involve a large number of spacetime points are strongly suppressed. As the size of the fluctuation decreases, the probability for events in which three or more spacetime points are correlated increases, in close analogy with the kinetic theory of gases (see Garay, 1999).

In the weak-coupling approximation, i.e., up to second order in the expansion parameter, the trilocal and higher effective interactions do not contribute. The terms corresponding to $N = 0, 1$ are local and can be absorbed in the bare action. Consequently, we can write the influence action $\mathcal{W}$ as a bilocal whose most general form is (Feynman and Hibbs, 1965)

$$\mathcal{W}[\phi, \varphi; t] = -\frac{1}{2}\int_0^t ds \int_0^s ds' \{h_i[\phi; s] - h_i[\varphi; s]\} \times \{\upsilon^{ij}(s - s')h_j[\phi; s'] - \upsilon^{ij}(s - s')^* h_j[\varphi; s']\},$$

where we have renamed $\upsilon^{ij}(s, s')$ as $\upsilon^{ij}(s - s')$, and without loss of generality we have set $s > s'$. This complex coefficient is Hermitian in the pair of indices ij and depends on the spatial positions $x_{\underline{i}}$ and $x_{\underline{j}}$ only through the relative distance $|x_{\underline{i}} - x_{\underline{j}}|$. It is of order $e^{-S(r)}$ and is concentrated within a spacetime region of size r.

Let us now decompose $\upsilon^{ij}(\tau)$ in terms of its real and imaginary parts as

$$\upsilon^{ij}(\tau) = c^{ij}(\tau) + i\, \dot{f}^{ij}(\tau),$$

where $c^{ij}(\tau)$ and $f^{ij}(\tau)$ are real and symmetric, and the overdot denotes time derivative. The imaginary part is antisymmetric in the exchange of i, τ and $j, -\tau$ and has been written as a time derivative for convenience, since this choice does

not involve any restriction. The f term can then be integrated by parts to obtain

$$\begin{aligned}\mathcal{W}[\phi,\varphi;t] = &-\frac{1}{2}\int_0^t ds\int_0^s ds'\, c^{ij}(s-s')\{h_i[\phi;s]-h_i[\varphi;s]\}\{h_j[\phi;s'] \\ &- h_j[\varphi;s']\} \\ &-\frac{i}{2}\int_0^t ds\int_0^s ds'\, f^{ij}(s-s')\{h_i[\phi;s]-h_i[\varphi;s]\}\{\dot{h}_j[\phi;s'] \\ &+ \dot{h}_j[\varphi;s']\}.\end{aligned}$$

In this integration, we have ignored surface terms that contribute, at most, to a finite renormalization of the bare low-energy Hamiltonian.

The functions $f^{ij}(\tau)$ and $c^{ij}(\tau)$ characterize spacetime foam in our effective description but, under fairly general assumptions, the characterization can be carried out by a smaller set of independent functions. Indeed, Lorentz invariance and spatial homogeneity, together with a kind of equity principle by which spacetime foam produces interactions whose intensity does not depend on the pair of interactions h_i itself but on its independent components for each mode, imply that $f^{ij}(\tau)$ and $c^{ij}(\tau)$ must have the form

$$f^{ij}(\tau) = \int_0^\infty d\omega\, G^{ij}(\omega)\,\cos(\omega\tau), \tag{3}$$

$$c^{ij}(\tau) = \int_0^\infty d\omega\, g(\omega) G^{ij}(\omega)\,\cos(\omega\tau), \tag{4}$$

where

$$G^{ij}(\omega) = 8\pi\,\frac{\sin(\omega|x_{\underline{i}}-x_{\underline{j}}|)}{\omega|x_{\underline{i}}-x_{\underline{j}}|}\chi^{\underline{i}}(\omega)\chi^{\underline{j}}(\omega),$$

and $g(\omega)$ is a function that, together with $\chi^{\underline{i}}(\omega)$, fully characterize spacetime foam under these assumptions. The functions $\chi^{\underline{i}}(\omega)$ can be interpreted as the spectral effective couplings between spacetime foam and low-energy fields. Since $v^{ij}(\tau)$ is of order $e^{-S(r)}$ and is concentrated in a region of linear size r, the couplings $\chi^{\underline{i}}(\omega)$ will have dimensions of length, will be of order $e^{-S(r)/2}r$, and will induce a significant interaction for all frequencies ω up to the natural cutoff r^{-1}. On the other hand, the function $g(\omega)$ has dimensions of inverse length and must be of order r^{-1}. Actually, this function must be almost flat in the frequency range $(0, r^{-1})$ to ensure that all the modes contribute significantly to all bilocal interactions. As we will see, the function $g(\omega)$ also admits a straightforward interpretation in terms of the mean occupation number for the mode of frequency ω.

If we restrict to the case in which $f^{ij}(\tau)$ vanishes, i.e., $v^{ij}(\tau) = c^{ij}(\tau)$, then the influence functional $\exp\mathcal{W}_c$ is the characteristic functional of a Gaussian

probability functional distribution, i.e., it can be written as

$$\exp \mathcal{W}_c[\phi, \varphi; t] = \int \mathcal{D}\alpha \, e^{-\frac{1}{2}\int_0^t ds \int_0^s ds' \, \gamma_{ij}(s-s')\alpha^i(s)\alpha^j(s')} \, e^{i\int_0^t ds\, \alpha^i(s)\{h_i[\phi;s]-h_i[\varphi;s]\}}.$$

Here, the continuous matrix $\gamma_{ij}(s-s')$ is the inverse of $c^{ij}(s-s')$, i.e.,

$$\int ds'' \, \gamma_{ik}(s-s'')c^{kj}(s''-s') = \delta_i^j \delta(s-s').$$

Then, in this case, the propagator $\$(t)$ has the form

$$\$(t) = \int \mathcal{D}\alpha \, P[\alpha]\$_\alpha(t),$$

where $\$_\alpha(t)$ is just a factorizable propagator associated with unitary evolution governed by the action $S_0 + \int \alpha^i h_i$ and

$$P[\alpha] = e^{-\frac{1}{2}\int_0^t ds \int_0^s ds' \, \gamma_{ij}(s-s')\alpha^i(s)\alpha^j(s')}.$$

Therefore, $\$(t)$ is just the average with Gaussian weight $P[\alpha]$ of the unitary propagator $\$_\alpha(t)$.

Note that the quadratic character of the distribution for the fields α^i is a consequence of the weak-coupling approximation, which keeps only the bilocal term in the action. Higher-order terms would introduce deviations from this noise distribution. The nonunitary nature of the bilocal interaction has been encoded inside the fields α^i, so that, when insisting on writing the system in terms of unitary evolution operators, an additional sum over the part of the system that is unknown naturally appears. Note also that we have a different field α^i for each kind of interaction h_i. Thus, we have transferred the nonlocality of the low-energy field ϕ to the set of fields α^i, which are nontrivially coupled to it and that represent spacetime foam.

3.2. Semiclassical Diffusion

We can see that the limit of vanishing $f^{ij}(\tau)$, with nonzero $c^{ij}(\tau)$ (and therefore real $v^{ij}(\tau)$), is a kind of semiclassical approximation since, in this limit, one ignores the quantum nature of the gravitational fluctuations. Indeed, the fields α^i represent spacetime foam but, as we have seen, the path integral for the whole system does not contain any trace of the dynamical character of the fields α^i. It just contains a Gaussian probability distribution for them. The path integral above can then be interpreted as a Gaussian average over the classical noise sources α^i. Classicality here means that we can keep the sources α^i fixed, ignoring the noise commutation relations, and, at the end of the calculations, we just average over them.

The low-energy density matrix ρ then satisfies the following master equation (Garay, 1998a,b, 1999)

$$\dot{\rho} = -i\big[H_0, \rho\big] - \int_0^\infty d\tau\, c^{ij}(\tau)\big[h_i, \big[h_j^{\rm I}(-\tau), \rho\big]\big],$$

where $h_j^{\rm I}(-\tau) = e^{-iH_0\tau} h_j\, e^{iH_0\tau}$. Since $e^{iH_0\tau} = 1 + O(\tau/l)$, the final form of the master equation for a low-energy system subject to gravitational fluctuations treated as a classical environment and at zeroth order in r/l (the effect of higher order terms in r/l will be thoroughly studied together with the quantum effects) is

$$\dot{\rho} = -i[H_0, \rho] - \int_0^\infty d\tau\, c^{ij}(\tau)[h_i, [h_j, \rho]].$$

(For similar approaches yielding this type of master equation see also Banks *et al.*, 1984; Diósi, 1987; Percival, 1995).

The first term gives the low-energy Hamiltonian evolution that would also be present in the absence of fluctuations. The second term is a diffusion term that will be responsible for the loss of coherence (and the subsequent increase of entropy). It is a direct consequence of the foam-like structure of spacetime and the related existence of a minimum length. Note there is no dissipation term. This term is usually present in order to preserve the commutation relations under time evolution. However, we have considered the classical noise limit, i.e., the noise α has been considered as a classical source and the commutation relations are automatically preserved. We will see that the dissipation term, apart from being of quantum origin, is r/l times smaller than the diffusion term and we have only considered the zeroth order approximation in r/l.

The characteristic decoherence time $\tau_{\rm d}$ induced by the diffusion term can be easily calculated. Indeed, the interaction Hamiltonian density h_i is of order $\ell_*^{-4}(\ell_*/l)^{2n_{\underline{i}}(1+s_{\underline{i}})}$ and $c^{ij}(\tau)$ is of order $e^{-S(r)}$. Furthermore, the diffusion term contains one integral over time and two integrals over spatial positions. The integral over time and the one over relative spatial positions provide a factor r^4, since $c^{ij}(\tau)$ is different from zero only in a spacetime region of size r^4, and the remaining integral over global spatial positions provides a factor l^3, the typical low-energy spatial volume. Putting everything together, we see that the diffusion term is of order $l^{-1}\epsilon^2 \sum_{\underline{ij}} (\ell_*/l)^{\eta_{\underline{i}}+\eta_{\underline{j}}}$, with $\eta_{\underline{i}} = 2n_{\underline{i}}(1 + s_{\underline{i}}) - 2$ and $\epsilon = e^{-S(r)/2}(r/\ell_*)^2$. This quantity defines the inverse of the decoherence time $\tau_{\rm d}$. Therefore, the ratio between the decoherence time $\tau_{\rm d}$ and the low-energy length scale l is

$$\tau_{\rm d}/l \sim \epsilon^{-2}\left[\sum_{\underline{ij}} (\ell_*/l)^{\eta_{\underline{i}}+\eta_{\underline{j}}}\right]^{-1}.$$

Because of the exponential factor in ϵ, only the gravitational fluctuations whose size is very close to Planck length will give a sufficiently small decoherence time.

Slightly larger fluctuations will have a very small effect on the unitarity of the effective theory.

3.3. Effective Quantum Bath

As we have briefly mentioned before, considering that the coefficients v^{ij} are real amounts to ignore the quantum dynamical nature of spacetime foam, paying attention only to its statistical properties. In what follows, we will study these quantum effects and show that spacetime foam can be effectively described in terms of a quantum thermal bath with a nearly Planckian temperature that has a weak interaction with low-energy fields. As a consequence, other effects, apart from loss of coherence, such as Lamb and Stark transition-frequency shifts, and quantum damping, characteristic of systems in a quantum environment (Gardiner, 1991), naturally appear as low-energy predictions of this model (Garay, 1998a,b, 1999).

Let us consider a Hamiltonian of the form

$$H = H_0 + H_{\text{int}} + H_{\text{b}}.$$

H_0 is the bare Hamiltonian that represents the low-energy fields and H_{b} is the Hamiltonian of a bath that, for simplicity, will be represented by a real massless scalar field. The interaction Hamiltonian will be of the form $H_{\text{int}} = \xi^i h_i$, where the noise operators ξ^i are given by

$$\xi^i(t) = i \int dk\, \sqrt{\omega}\, \underline{\chi^i}(\omega) \left[a^\dagger(k)\, e^{i(\omega t - kx)} - a(k)\, e^{-i(\omega t - kx)} \right],$$

where $\omega = \sqrt{k^2}$, and a and $a^\dagger$ are, respectively, the annihilation and creation operators associated with the bath.

The influence functional in this case has the form (Fenyman and Hibbs, 1965)

$$e^{\mathcal{W}[\phi,\varphi;t]} = \int \mathcal{D}q'\, \mathcal{D}Q'\, \rho_{\text{b}}[q',\, Q';0] \int \mathcal{D}q\, \mathcal{D}Q\, e^{i S_{\text{b}}[q;t] - i S_{\text{b}}[Q;t]} \times e^{i S_{\text{int}}[\phi,q;t] - i S_{\text{int}}[\varphi,Q;t]},$$

where these path integrals are performed over paths $q(s)$ and $Q(s)$ such that at the initial time match the values q' and Q' and S_{b} is the action of the bath.

If we assume that the bath is in a stationary, homogeneous, and isotropic state, this influence functional can be computed to yield an influence action $\mathcal{W}$ of the form discussed above. Furthermore, for a thermal state with temperature $T \sim 1/r$, the function $g(\omega)$ has the form

$$g(\omega) = \omega[N(\omega) + 1/2],$$

where $N(\omega) = [\exp(\omega/T) - 1]^{-1}$ is the mean occupation number of the quantum thermal bath corresponding to the frequency ω. Recall that the functions $G^{ij}(\omega)$

and, hence, $f^{ij}(\tau)$ are uniquely determined by the couplings $\chi^{i}(\omega)$. In particular, they are completely independent of the state of the bath or the system. All the relevant information about the bath is encoded in the function $g(\omega)$.

With this procedure, we see that spacetime foam can be represented by a quantum bath determined by $g(\omega)$ that interacts with the low-energy fields by means of the couplings $\chi^{i}(\omega)$ which characterize spacetime foam, in the sense that both systems produce the same low-energy effects because they are described by the same influence action $\mathcal{W}$.

This model that we have proposed is particularly suited to the study of low-energy effects produced by simply connected topology fluctuations such as closed loops of virtual black holes (Hawking, 1996). Virtual black holes will not obey classical equations of motion but will appear as quantum fluctuations of spacetime and thus will become part of the spacetime foam as we have discussed. Particles could fall into these black holes and be reemitted. The scattering amplitudes of these processes (Hawking, 1996; Hawking and Ross, 1997) could be interpreted as being produced by nonlocal effective interactions that would take place inside the fluctuations and the influence functional obtained above could then be interpreted as providing the evolution of the low-energy density matrix in the presence of a bath of ubiquitous quantum topological fluctuations of the virtual-black-hole type.

3.4. Master Equation

As we have already mentioned, from the influence functional obtained in the previous section, we can obtain the master equation satisfied by the low-energy density matrix, although here we will follow a different procedure: We will derive the master equation in the canonical formalism from von Neumann equation for the joint system of the low-energy fields plus the effective quantum bath coupled to them that accounts for the effects of spacetime foam.

It is easy to see that the function $f^{ij}(\tau)$ given in Eq. (3) determines the commutation relations at different times of the noise variables. Indeed, taking into account the commutation relations for the annihilation and creation operators a and $a^{\dagger}$, we obtain by direct calculation the relation

$$[\xi^{i}(t), \xi^{j}(t')] = i\,\frac{d}{dt} f^{ij}(t - t').$$

Similarly, the function $c^{ij}(\tau)$ of Eq. (4) determines the average (defined as the trace over the bath) of the anticommutator of the noise variables,

$$\frac{1}{2}\langle [\xi^{i}(t), \xi^{j}(t')]_{+} \rangle = c^{ij}(t - t'),$$

provided that the bath is in a stationary, homogeneous, and isotropic state

determined by $g(\omega)$, i.e.,

$$\langle a(k)\rangle = 0, \quad \langle a(k)a(k')\rangle = 0, \quad \langle a^\dagger(k)a(k')\rangle = [g(\omega)/\omega - 1/2]\delta(k-k').$$

We are now ready to write down the master equation for the low-energy density matrix. We will describe the whole system (low-energy field and bath) by a density matrix $\rho_{\rm T}(t)$. We will assume that, initially, the low energy fields and the bath are independent, i.e., that at the time $t = 0$, $\rho_{\rm T}(0) = \rho(0) \otimes \rho_{\rm b}$. If the low-energy fields and the bath do not decouple at any time, an extra renormalization term should be added to the Hamiltonian. In the interaction picture, the density matrix has the form $\rho_{\rm T}^{\rm I}(t) = U^\dagger(t)\rho_{\rm T}(t)U(t)$, with $U(t) = e^{-iH_0 t}e^{-iH_{\rm b}t}$, and obeys the equation of motion

$$\dot\rho_{\rm T}^{\rm I}(t) = -i\big[\xi^i(t)h_i^{\rm I}(t),\, \rho_{\rm T}^{\rm I}(t)\big],$$

where $\xi^i(t) = U^\dagger(t)\xi^i U(t)$ and $h_i^{\rm I}(t) = U^\dagger(t)h_i U(t)$. Integrating this evolution equation and introducing the result back into it, tracing over the variables of the bath, defining $\rho^{\rm I}(t) \equiv {\rm tr}_{\rm b}[\rho_{\rm T}^{\rm I}(t)]$, and noting that ${\rm tr}_{\rm b}[\xi^i(t)h_i^{\rm I}(t)\rho_{\rm T}^{\rm I}(t_0)] = 0$, we obtain

$$\dot\rho^{\rm I}(t) = -\int_{t_0}^{t} dt'\,{\rm tr}_{\rm b}\left\{\big[\xi^i(t)h_i^{\rm I}(t),\, \big[\xi^j(t')h_j^{\rm I}(t'),\, \rho_{\rm T}^{\rm I}(t')\big]\big]\right\}.$$

In the weak-coupling approximation, which implies that $\xi^i h_i$ is much smaller than H_0 and $H_{\rm b}$ (this is justified since it is of order ϵ), we assume that the bath density matrix does not change because of the interaction, so that $\rho_{\rm T}^{\rm I}(t) = \rho^{\rm I}(t) \otimes \rho_{\rm b}$. The error introduced by this substitution is of order ϵ and ignoring it in the master equation amounts to keep terms only up to second order in this parameter. Since $\big[\xi^i(t),\, h_j^{\rm I}(t')\big] = 0$ because $\big[\xi^i,\, h_j\big] = 0$, the right-hand side of this equation can be written in the following way

$$-\int_0^t dt'\Big\{c^{ij}(t-t')\big[h_i^{\rm I}(t),\, \big[h_j^{\rm I}(t'),\, \rho^{\rm I}(t')\big]\big] + \frac{i}{2}\dot f^{ij}(t-t')\big[h_i^{\rm I}(t),\, \big[h_j^{\rm I}(t'),\, \rho^{\rm I}(t')\big]_+\big]\Big\}.$$

The Markov approximation allows the substitution of $\rho^{\rm I}(t')$ by $\rho^{\rm I}(t)$ in the master equation because the integral over t' will get a significant contribution from times t' that are close to t due to the factors $\dot f^{ij}(t-t')$ and $c^{ij}(t-t')$ and because, in this interval of time, the density matrix $\rho^{\rm I}$ will not change significantly. Indeed, the typical evolution time of $\rho^{\rm I}$ is the low-energy time scale l, which will be much larger than the time scale r associated with the bath. If we perform a change of the integration variable from t' to $\tau = t - t'$, write

$$\rho^{\rm I}(t') = \rho^{\rm I}(t-\tau) = \rho^{\rm I}(t) - \tau\dot\rho^{\rm I}(t) + O(\tau^2),$$

and introduce this expression in the master equation above, we easily see that the error introduced by the Markovian approximation is of order ϵ^2, i.e., it amounts ignore a term of order ϵ^4. The upper integration limit t in both integrals can be substituted by ∞ for evolution times t much larger than the correlation time r, because of the factors $\dot{f}^{ij}(\tau)$ and $c^{ij}(\tau)$ that vanish for $\tau > r$.

Then, after an integration by parts of the f term, and transforming the resulting master equation back to the Schrödinger picture we obtain

$$\dot{\rho} = -i\big[H_0', \rho\big] - \frac{i}{2}\int_0^\infty d\tau\ f^{ij}(\tau)\big[h_i, \big[\dot{h}_j^{\mathrm{I}}(-\tau), \rho\big]_+\big] - \int_0^\infty d\tau\, c^{ij}(\tau) \times \big[h_i, \big[h_j^{\mathrm{I}}(-\tau), \rho\big]\big],$$

where $H_0' = H_0 - \frac{1}{2} f^{ij}(0) h_i h_j$ is just the original low-energy Hamiltonian plus a finite renormalization originated in the integration by parts of the f term. It can be checked that the low-energy density matrix $\rho(t)$ obtained by means of the influence action $\mathcal{W}$ is indeed a solution of this master equation.

3.5. Low-Energy Effects

Let us now analyze the general master equation, valid up to second order in ϵ, that takes into account the quantum nature of the gravitational fluctuations. These contributions will be fairly small in the low-energy regime, but may provide interesting information about the higher-energy regimes in which l may be of the order of a few Planck lengths and for which the weak-coupling approximation is still valid. In order to see these contributions explicitly, let us further elaborate the master equation. In terms of the operator L_0 defined as $L_0 \cdot A = \big[H_0, A\big]$ acting of any low-energy operator A, the time-dependent interaction $h_j^{\mathrm{I}}(-\tau)$ can be written as

$$h_j^{\mathrm{I}}(-\tau) = e^{-iL_0\tau} h_j.$$

The interaction h_j can be expanded in eigenoperators $h_{j\Omega}^{\pm}$ of the operator L_0, i.e.,

$$h_j = \int d\mu_\Omega \big(h_{j\Omega}^{+} + h_{j\Omega}^{-}\big),$$

with $L_0 \cdot h_{j\Omega}^{\pm} = \pm\Omega h_{j\Omega}^{\pm}$ and $d\mu_\Omega$ being an appropriate spectral measure, which is naturally cut off around the low-energy scale l^{-1}. This expansion always exists provided that the eigenstates of H_0 form a complete set. Then, $h_j^{\mathrm{I}}(-\tau)$ can be written as

$$h_j^{\mathrm{I}}(-\tau) = \int d\mu_\Omega (e^{-i\Omega\tau} h_{j\Omega}^{+} + e^{i\Omega\tau} h_{j\Omega}^{-}).$$

It is also convenient to define the new interaction operators for each low-energy frequency Ω

$$h^1_{j\Omega} = h^+_{j\Omega} - h^-_{j\Omega}, \quad h^2_{j\Omega} = h^+_{j\Omega} + h^-_{j\Omega}.$$

The quantum noise effects are reflected in the master equation through the term proportional to $f^{ij}(\tau)$ and the term proportional to $c^{ij}(\tau)$, both of them integrated over $\tau \in (0, \infty)$. Because of these incomplete integrals, each term provides two different kinds of contributions whose origin can be traced back to the well-know formula

$$\int_0^\infty d\tau\, e^{i\omega\tau} = \pi\delta(\omega) + \mathcal{P}(i/\omega),$$

where $\mathcal{P}$ is the Cauchy principal part (Reed and Simon, 1972).

The master equation can then be written in the following form

$$\dot{\rho} = -(iL'_0 + L_{\text{diss}} + L_{\text{diff}} + iL_{\text{s-l}}) \cdot \rho,$$

where the meaning of the different terms are explained in what follows.

The first term $-iL'_0 \cdot \rho$, with $L'_0 \cdot \rho = [H'_0, \rho]$, is responsible for the renormalized low-energy Hamiltonian evolution. The renormalization term is of order ε^2 as compared with the low-energy Hamiltonian H_0, where $\varepsilon^2 = \epsilon^2 \sum_{\underline{i}\,\underline{j}} (\ell_*/l)^{\eta_{\underline{i}}+\eta_{\underline{j}}}$ and, remember, $\eta_{\underline{i}} = 2n_{\underline{i}}(1 + s_{\underline{i}}) - 2$ is a parameter specific to each kind of interaction term h_i.

The dissipation term

$$L_{\text{diss}} \cdot \rho = \frac{\pi}{4} \int d\mu_\Omega\, \Omega G^{ij}(\Omega)\big[h_i, \big[h^1_{j\Omega}, \rho\big]_+\big]$$

is necessary for the preservation in time of the low-energy commutators in the presence of quantum noise. As we have seen, it is proportional to the commutator between the noise creation and annihilation operators and, therefore, vanishes in the classical noise limit. Its size is of order $\varepsilon^2 r/l^2$.

The diffusion process is governed by

$$L_{\text{diff}} \cdot \rho = \frac{\pi}{2} \int d\mu_\Omega\, g(\Omega) G^{ij}(\Omega)\big[h_i, \big[h^2_{j\Omega}, \rho\big]\big],$$

which is of order ε^2/l.

The next term provides an energy shift that can be interpreted as a mixture of a gravitational ac Stark effect and a Lamb shift by comparison with its quantum optics analog (Gardiner, 1991). Its expression is

$$L_{\text{s-l}} = -\int d\mu_\Omega\, \mathcal{P} \int_0^\infty d\omega \frac{\Omega}{\omega^2 - \Omega^2} G^{ij}(\omega) \left\{ g(\omega)\big[h_i, \big[h^1_{j\Omega}, \rho\big]\big] + \frac{\Omega}{2}\big[h_i, \big[h^2_{j\Omega}, \rho\big]_+\big] \right\}.$$

The second term is of order $\varepsilon^2 r^2/l^3$, which is fairly small. However, the first term will provide a significant contribution of order $\varepsilon^2 r/l^2[\ln(l/r)+1]$. This logarithmic dependence on the relative scale is indeed characteristic of the Lamb shift (Gardiner, 1991; Itzykson and Zuber, 1985). As we have argued the function $g(\omega)$ must be fairly flat in the whole range of frequencies up to the cutoff $1/r$ and be of order $1/r$ in order to reproduce the appropriate correlations $c^{ij}(\tau)$. A thermal bath, for instance, produces a function $g(\omega)$ with the desired characteristics, at least at the level of approximation that we are considering. In this specific case, it can be seen that the logarithmic contribution to the energy shift is not present and it would only appear in the zero temperature limit. However, since we are modeling spacetime foam with this thermal bath, the effective temperature is $1/r$, which is close to Planck scale and certainly far from zero. From the practical point of view, the presence or not of this logarithmic contribution is at most an order of magnitude larger than the standard one and, therefore, it does not significantly affect the results. Almost any other state of the bath with a more or less uniform frequency distribution will contain such logarithmic contribution.

As a summary, the f term provides a dissipation part, necessary for the preservation of commutators, and a fairly small contribution to what can be interpreted as a gravitational Lamb shift. On the other hand, the c term gives rise to a diffusion term and a shift in the oscillation frequencies of the low-energy fields that can be interpreted as a mixture of a gravitational Stark effect and a Lamb shift. The size of these effects, compared with the bare evolution, are the following: the diffusion term is of order ε^2; the damping term is smaller by a factor r/l, and the combined effect of the Stark and Lamb shifts is of order $(r/l)[\ln(l/r)+1]$ as compared with the diffusion term. Note that the quantum effects induced by spacetime foam become relevant as the low-energy length scale l decreases, as we see from the fact that these effects depend on the ratio r/l, while, in this situation, the diffusion process becomes faster, except for the mass of scalars, which always decoheres in a time scale that is close to the low-energy evolution time.

4. CONCLUSIONS

In our study of the evolution of quantum systems according to real clocks, which are necessarily subject to errors, we have first established a stochastic characterization of good real clocks. Using this description, we have derived a master equation for the quantum evolution in real clock time and we have also found its general solution on the basis of energy eigenstates. The stochastic features of good real clocks and their effects on the quantum evolution can be equivalently described by means of interactions that are nonlocal in time. They can also be effectively modeled by a quantum thermal bath. The master equation exhibits a diffusion term, which is responsible for the loss of coherence of most initial states.

Finally, we have analyzed the evolution of classical systems according to real clocks and reached analogous conclusions.

The third law of thermodynamics and the quantum fluctuations prevent real clocks from being perfectly accurate. This suggests that, strictly speaking, the Schrödinger unitary evolution equation is just an excellent approximation valid for sufficiently short periods of time and that should be substituted, along the lines proposed in this paper, by a diffusive master equation in more general situations. This adds a random aspect to the evolution of quantum systems. Indeed, coherence is progressively lost until we reach the period of applicability of the clock and, after that time, unpredictability sets in, as we have seen. Even perfectly isolated systems will suffer loss of coherence because of the fluctuations of the real clock and will appear as effectively coupled to a reservoir.

Quantum fluctuations of the gravitational field on the other hand may well give rise to the existence of a minimum length in the Planck scale. This can be seen, for instance, by making use of the fact that measurements and vacuum fluctuations of the gravitational field are extended both in space and time and can therefore be treated with the techniques employed for continuous measurements, in particular the action uncertainty principle (Mensky, 1992). The existence of this resolution limit spoils the metric structure of spacetime at the Planck scales and opens a doorway to nontrivial topologies, which will not only contribute to the path integral formulation but will also dominate the Planck scale physics thus endowing spacetime with a foam-like structure with very complicated topology. Indeed, at the Planck scale, both the partition function and the density of topologies seem to receive the dominant contribution from topological configurations with very high Betti numbers (Carlip, 1998; Hawking, 1978).

Spacetime foam may leave its imprint in the low-energy physics and it seems to induce loss of coherence in the low-energy quantum fields that propagate on it as well as mode-dependent energy shifts. In order to study some of these effects in more detail, we have built an effective theory in which spacetime foam has been substituted by a fixed classical background plus nonlocal interactions between the low-energy fields confined to bounded spacetime regions of nearly Planck size. In the weak-coupling approximation, these nonlocal interactions become bilocal. The low-energy evolution is nonunitary because of the absence of a nonvanishing time-like Hamiltonian vector field. The nonunitarity of the bilocal interaction can be encoded in a quantum noise source locally coupled to the low-energy fields. From the form of the influence functional that accounts for the interaction with spacetime foam, we have derived a master equation for the evolution of the low-energy fields, which contains a diffusion term, a damping term, and energy shifts that can be interpreted as gravitational Lamb and Stark effects.

As we have seen, there exist strong similarities between the evolution in spacetime foam and that in quantum mechanics with real clocks. In both cases, the fluctuations are described statistically and induce loss of coherence. However, there

are some major differences. In the case of real clocks, the diffusion term contains only the Hamiltonian of the system while, in the spacetime foam analysis, a plethora of interactions appeared. Closely related to this, fluctuations of the real clock affect in very similar ways to both classical and quantum evolution; this is not the case in spacetime foam. The origin of these differences is the nature of the fluctuations that we are considering and, more specifically, the existence or not of horizons. Indeed, when studying real clocks, we have ensured that they satisfied Galilean causality, i.e., that the real-time parameter always grows as compared with the ideal time, so that no closed time-like curves are allowed in Galilean spacetime, whichever clock we are using. This requirement is in sharp contrast with the situation that we find in spacetime foam, where we have to consider topological fluctuations that contain horizons (virtual black holes, time machines, etc.). Scattering processes in a spacetime with horizons are necessarily of quantum nature. A classical scattering process in the presence of these horizons would inevitably lead to loss of probability because of the particles that would fall inside the horizons and would never come out to the asymptotic region.

In other words, the underlying dynamics is completely different in both cases. Spacetime foam provides a non-Hamiltonian dynamics since the underlying manifold is not globally hyperbolic. On the other hand, in the case of quantum mechanics according to clocks subject to small errors, the underlying evolution is purely Hamiltonian, although the effective one is an average over all possible Hamiltonian evolutions and becomes nonunitary.

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